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CALCULUS SUPPLEMENT

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FRANKLIN, MACNUTT AND CHARLES

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PREFACE.

1. Our chief endeavor in this elementary treatise on calculus has been to develop the subject as simply and directly as possible, and to make our discussions vividly concrete and suggestive. Many of the most serious difficulties to the beginner in his study of calculus come from the fact that his attention may not be directed to the simple arithmetical principles of the subject; therefore an elementary treatise on calculus should be developed to some extent as a phase of an old fashioned branch of arithmetic, namely, mental arithmetic. A good example of this is shown in Article 31.

2. We fully appreciate the importance of extensive practice in the handling of algebraic transformations. Therefore we have given an adequate collection of formal problems in differentiation and integration.

3. We are convinced, however, that it is a pedagogical mistake, in a text book on calculus for young men, to break the thread of the textual discussion by unnecessary algebraic developments and by large and frequent groups of purely formal problems. We believe indeed that the proverbial unintelligibility of calculus is to a very great extent a psychological consequence of this almost universal and really hideous feature. Therefore we have arranged the greater portion of our formal problems in an appendix.

4. Nearly every elementary science text in existence carries a false suggestion of completeness and finality, and there are two things which young men should understand in connection with their study of the mathematical sciences. The first is that such study is exacting beyond all compromise, involving as it does a degree of constraint which it is beyond the power of any teacher

greatly to mitigate. And the second is that the completest science stands abashed before the infinitely complicated and fluid array of phenomena of the material world, except only in the assurance which method gives. We hope that this elementary treatise on calculus may prove to be sufficiently definite and exacting to be useful; next to this there is nothing we could wish to have more in evidence from beginning to end than its incompleteness.

In order to emphasize the incomplete character of this text we have introduced references to more complete treatises throughout the text and we have given in Appendix *C* a carefully selected list of treatises on mathematics and on mathematical physics. Teachers who use this text should, we believe, direct the students' attention to this appendix.

Nearly all writers on applied mathematics follow the procedure which is outlined at the end of Art. 23. We follow this procedure to some extent in Arts. 49, 50 and 51, to some extent in Chapter IV, and throughout Chapter V. This procedure does, no doubt, weaken the emphasis upon the important steps in any argument leading to a limiting ratio in calculus; but it favors simplicity and directness of speech and it means an all-important gain in concreteness.

Inexactness and false suggestion should never be tolerated in an elementary treatise on mathematical science; but to insist upon completeness of symbolism in a calculus argument is not the same thing as to insist upon rigor. This is clearly illustrated in Arts. 31 and 34 where the symbolism is left somewhat incomplete in order not to obscure the underlying ideas. Many ideas in the mathematical sciences are purely mechanical or purely arithmetical, and an exact verbal statement of any mechanical or arithmetical relation is necessarily a labored statement. Once a person gets such an idea clearly in mind, the exact verbal statement appears in its true character as almost grotesquely cumbersome! For the great majority of men and women, language seems still to be a vehicle for sentiment, as it

was, no doubt, in the beginning, and the greatest problem in the teaching of the sciences is how to qualify young men and women for the increasingly important impersonal uses of speech, especially of speech as it stands on the printed page. If this be true, what a mistake it is to burden an elementary mathematical treatise with unnecessary formalism!

D'Alembert once said to a young mathematical student who came to him in perplexity "Go ahead, young man, go ahead! Conviction will come to you later." In the preface to our first edition we adopted D'Alembert's advice as our own, but we are now inclined to turn it around, and say to the perplexed student Go back, young man, go back! Understanding comes from an adequate consideration of the simple beginnings of a subject! It is, indeed, a bad rule which does not work both ways. And this rule of D'Alembert's is certainly not a bad rule.

"The absolute requisites for the study of this work are: a knowledge of elementary algebra to the binomial theorem (according to the usual arrangement), plane and solid geometry, trigonometry, and the most simple parts of the usual applications of algebra to geometry."

This was said by De Morgan in the preface to his great treatise on *Differential and Integral Calculus* (London, 1842), in comparison with which this book is a primer.

FRANKLIN, MACNUTT and CHARLES.

CHAPTER I.

GENERAL SURVEY OF DIFFERENTIAL AND INTEGRAL CALCULUS.

1. Constant quantities and variable quantities.—Elementary algebra deals only with quantities which do not change in value (*constant quantities*). For example let x be the quantity which subtracted from 12 leaves a remainder equal to $\frac{1}{2}x$. That is, $12 - x = \frac{1}{2}x$, whence $x = 8$. The quantity x has the same value throughout this discussion and it is therefore called a constant quantity. Calculus is a branch of algebra and it deals not only with constant quantities but also and especially with quantities which change in value (*variable quantities*). Thus one of the simplest problems of calculus is to consider how rapidly x^2 grows in value when x is imagined to grow at a definite rate.

Variable quantities fall into three fairly distinct classes, namely: (*a*) Quantities which vary in time, (*b*) quantities which vary in space, and (*c*) quantities which are arbitrarily assumed to vary. An example of the first class is the varying temperature of the air *at a given place* as the seasons come and go. An example of the second class is the varying temperature of the air from place to place *at a given time.*

Variables of the first and second classes are sometimes called *natural variables* to distinguish them from variables of the third class, examples of which are considered in Arts. 10 and 11.

2. Value at a given instant. Value at a given point.—Let y represent the amount of water in a pail into which a small stream of water is flowing. Evidently y is a changing quantity. If the flow of water were to be shut off at *any given instant* there would be a *definite quantity* of water left in the pail, and, therefore, there is a definite amount of water in the pail at each instant even while

the inflow continues. *Any changing quantity y has a definite value at each instant.**

A quantity which varies in space has a definite value at each point.* Thus everyone understands that there is a definite temperature at Philadelphia although it may be different from the temperature at New York or Washington. An iron rod with one end in the fire has a definite temperature at each point.

3. Increments and decrements.—It is frequently desirable to consider an increase (or a decrease) in the value of a variable quantity. Such an increase is called an *increment.* A decrease is called a *decrement.* For example, let y be a variable quantity. An increment of y is usually represented by the symbol Δy. When Δy is negative it is a decrement. This symbol Δy *does NOT mean* Δ *multiplied by* y, **it is a single symbol,** and it may be read *increment-of-y* or *delta-y.* The latter is preferable because of its brevity. The Greek letter Δ used as a prefix always means *increment of.*

4. Rate of change of a quantity which varies in time.—Consider a pail into which water is flowing in a small stream. Let y be the amount of water in the pail. Then y is evidently a changing quantity. Consider the amount of water that flows into the pail during a short interval of time Δt; this amount of water is evidently the increment of y during the short interval of time, and it is to be represented by the symbol Δy. The quotient $\frac{\Delta y}{\Delta t}$ is called the *average rate of change of* y *during the interval* Δt.†

If the inflowing stream of water is constant, say, always 2 cubic inches of water per second, then if Δt is chosen smaller and smaller, the value of Δy will be *proportionately* smaller and smaller, and the value of the quotient $\frac{\Delta y}{\Delta t}$ will be exactly 2 cubic inches per second whatever the duration or length of the time-interval Δt.

* This may not be true of a discontinuous variable.

† Thus if y is the amount of money a man has saved, then, if he saves \$20 more during 30 days, we have $\Delta y = \$20$ and $\Delta t = 30$ days, and the average rate of saving during the 30 days will be $66\frac{2}{3}$ cents per day.

If the inflowing stream is variable, then the value of the quotient $\frac{\Delta y}{\Delta t}$ will not be the same for different values of Δt, but the amount of water which flows into the pail *during a very short interval of time* will be very nearly proportional to the interval. For example, a certain amount of water W flows into the pail during a particular one-thousandth of a second. Imagine this particular one-thousandth of a second to be divided in halves; then only a very little more than $\frac{1}{2}W$ will flow into the pail during one of the half-thousandths of a second and a very little less than $\frac{1}{2}W$ will flow into the pail during the other half-thousandth of a second. *If a shorter and shorter interval of time be taken* **the amount of inflow of water will be more and more nearly in EXACT proportion to the duration of the interval.** This means that the quotient $\frac{\Delta y}{\Delta t}$ approaches a definite limiting value as Δt and Δy both approach zero; and this limiting value of $\frac{\Delta y}{\Delta t}$ is called *the rate of change of y at a certain instant* **or the instantaneous rate of change of y.**

The limiting value of $\frac{\Delta y}{\Delta t}$ is always represented by $\frac{dy}{dt}$ which means *the rate of change of y at a given instant.*

Nearly everyone falls into the idea that such an expression as 10 feet per second means 10 feet of actual movement in an actual second of time, but a body moving at a velocity of 10 feet per second might not continue to move for a whole second or its velocity might change before a whole second has elasped. Three cubic inches per second is the same rate of inflow of water into a pail as 2,070 cubic yards per year, but to specify the *rate* of inflow in cubic yards per year does not mean that a whole cubic yard of water flows into the pail nor does it mean that the inflow continues for a year. A man does not need to work for a whole month to earn money at the *rate* of $60.00 per month, nor for a whole day to earn money at the *rate* of $2.00 per day. A falling body has

a velocity of 19,130,000 miles per century after it has been falling for one second, but to specify its velocity in miles per century does not mean that it falls as far as a mile nor that it continues to fall for a century! The units of length and time which appear in the specification of a velocity are completely swallowed up, as it were, in the idea of velocity; and the same thing is true of the specification of any rate.

5. Continuous variables and discontinuous variables.—A quantity which changes by sudden jumps is called a *discontinuous variable.* For example the amount of money one has is a discontinuous variable, because a debt is made on the instant that one decides to accept a purchase; that is to say, money is spent in lumps, and a lump of money is spent during an indefinitely short time. *The amount of money spent during an interval of time does NOT become more and more nearly proportional to the interval as the interval grows shorter and shorter,* and consequently it is meaningless to speak of *the rate of spending money at a given instant.* If y is a discontinuous variable and if the time-interval Δt happens to include a jump in the value of y, then the value of Δy remains finite as Δt approaches zero and the quotient $\frac{\Delta y}{\Delta t}$ approaches infinity. The rate of change of a discontinuous variable at a given instant is unthinkable.

The amount of water in a pail, as considered in Art. 4, is an example of what is called a *continuous variable.* If y is a continuous variable and if Δy is the increment of y during the time-interval Δt, then Δy becomes more and more nearly proportional to Δt as Δt approaches zero; that is to say, the quotient $\frac{\Delta y}{\Delta t}$ approaches a definite limiting value as Δt approaches zero, and this limiting value of $\frac{\Delta y}{\Delta t}$ is called the instantaneous rate of change of y.

In the study of calculus we deal almost exclusively with continuous variables.

6. Example showing determination of instantaneous rate of change.—Let us *assume* that the amount of water in the pail in the discussion of Art. 4 is proportional to the square of the time t which has elapsed since a chosen instant. Then we may write

$$y = kt^2 \tag{1}$$

where k is a constant. A moment later t has increased and of course y has increased also. Let us represent the new value of t by $t + \Delta t$ and the new value of y by $y + \Delta y$. Then, since equation (1) is assumed to be true for all values of t and y, we have

$$y + \Delta y = k(t + \Delta t)^2$$

or

$$y + \Delta y = kt^2 + 2kt \cdot \Delta t + k(\Delta t)^2 \tag{2}$$

Subtracting equation (1) from equation (2) member by member, we have:

$$\Delta y = 2kt \cdot \Delta t + k(\Delta t)^2 \tag{3}$$

Whence, dividing both members by Δt, we have:

$$\frac{\Delta y}{\Delta t} = 2kt + k \cdot \Delta t \tag{4}$$

Now it is evident that $k \cdot \Delta t$ approaches zero as Δt approaches zero, and therefore the limiting value of $\frac{\Delta y}{\Delta t}$ (as Δt approaches zero) is $2kt$. Hence, writing $\frac{dy}{dt}$ for the limiting value of $\frac{\Delta y}{\Delta t}$, we have:

$$\frac{dy}{dt} = 2kt \tag{5}$$

That is, the rate of change of y is at each instant equal to $2kt$.

Observation or thinking; which?—The discussion, in Art. 4, of the rate of increase of the amount of water in a pail presents a serious difficulty. How is one to know the increment of water which occurs during an indefinitely short interval of time? One

certainly cannot measure it. Observation and measurement are entirely useless in the consideration of an indefinitely small change which takes place during an indefinitely short interval of time. One cannot *observe* such things, one can only *think* about them. Indeed the whole matter at present under discussion is a matter of mathematical reasoning.

7. Gradient of a quantity which varies in space.—Consider an iron rod which stands with one end in a fire. Then the temperature of the rod varies from point to point along the rod. Consider two points very near together, let ΔT be their difference in temperature and let Δx be their distance apart. The quotient $\frac{\Delta T}{\Delta x}$ approaches a definite limiting value when Δx is chosen smaller and smaller, and this limiting value of $\frac{\Delta T}{\Delta x}$ is called the *temperature gradient* at the point (degrees per inch). In this example T varies *continuously* along the rod, it does not vary by jumps, otherwise the gradient of T at a point would be unthinkable.

The limiting value of the quotient $\frac{\Delta T}{\Delta x}$ is always represented by $\frac{dT}{dx}$, and it means the gradient of T at a given point.

Another quantity which varies from point to point in space is the pressure of the atmosphere; the higher one goes the less the pressure. Consider two points A and B, one of which is at a distance Δx above the other, and let Δp be the amount by which the pressure at A exceeds the pressure at B. Then $\frac{dp}{dx}$, the *limiting value of the quotient* $\frac{\Delta p}{\Delta x}$, is called the *pressure gradient* at A.* The pressure gradient of the atmosphere at sea level is usually about 2.3 pounds-per-square-inch per mile.

The word *gradient* as used in the above discussion comes from the word *grade* meaning the steepness of a slope. Thus if the

* Or at B; the two points A and B approach coincidence as Δx approaches zero.

slope of a hill at a given place rises Δh feet in a horizontal distance of Δx feet, then the limiting value of $\frac{\Delta h}{\Delta x}$ when Δx is made smaller and smaller is called the grade of the hill at the given place. The grade of a hill at a point might be one foot of rise per ten feet of horizontal distance, or ten feet of rise per hundred feet of horizontal distance, or 10 per cent. as it is usually expressed. It is evident that 10 feet of rise per 100 feet of horizontal distance does not mean necessarily 10 feet of actual rise in 100 feet of horizontal distance, because the slope may not be 100 feet long and the grade may vary from point to point.

Similarly a pressure gradient of 2.3 pounds-per-square-inch per mile, which is the upward gradient of atmospheric pressure at sea level, does not mean that the atmospheric pressure at a point one mile above sea level is 2.3 pounds per square inch less than at sea level, because the pressure gradient becomes less and less at points higher and higher above the sea. The units which appear in the specification of a grade or a gradient are completely swallowed up, as it were, in the idea of grade or gradient just as the units which appear in the specification of a rate are swallowed up in the idea of rate.

8. Example showing the determination of a gradient at a point. —Consider a metal bar AB, Fig. 1. Suppose the temperature

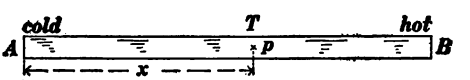

Fig. 1.

of the bar to be zero at the end A, and let us *assume* the temperature T at the point p to be:

$$T = kx^2 \qquad (1)$$

where x is the distance from A to p, and k is a constant. Under these conditions let it be required to find the temperature gradient at the point p. Now equation (1) is true for every value of x and T. Therefore writing $x + \Delta x$ for x and writing

$T + \Delta T$ for T we have

$$T + \Delta T = k(x + \Delta x)^2 \tag{2}$$

or

$$T + \Delta T = kx^2 + 2kx \cdot \Delta x + k(\Delta x)^2 \tag{3}$$

Subtracting equation (1) from equation (3) member by member, we have:

$$\Delta T = 2kx \cdot \Delta x + k(\Delta x)^2 \tag{4}$$

whence

$$\frac{\Delta T}{\Delta x} = 2kx + k \cdot \Delta x \tag{5}$$

Now $k \cdot \Delta x$ approaches zero as Δx approaches zero, and it is therefore evident that the limiting value of $\frac{\Delta T}{\Delta x}$ is $2kx$. Therefore, writing $\frac{dT}{dx}$ for the limiting value of $\frac{\Delta T}{\Delta x}$, we have:

$$\frac{dT}{dx} = 2kx \tag{6}$$

It is evident from this equation that the temperature gradient is zero at the end A of the bar where $x = 0$; and it is evident that the temperature gradient grows greater and greater (steeper and steeper) at points farther and farther from A where x is larger and larger.

PROBLEMS.

1. The amount of water in a pail is assumed to be proportional to the square of the time which has elasped since a chosen instant, according to equation (1) of Art. 6. It is evident from this equation that the amount of water in the pail is zero when $t = 0$. After the lapse of 20 seconds there is 600 cubic inches of water in the pail. Find the value of k and state the unit in terms of which k is expressed. Ans. 1.5 cubic inches per second per second.

2. Using the value of k from problem 1, find the rate at which water flows into the pail, (*a*) when $t = 5$ seconds, and (*b*) when $t = 20$ seconds. Ans. (*a*) 15 cubic inches per second, (*b*) 60 cubic inches per second.

3. Find the average rate of inflow of water into the pail during the time from $t = 0$ to $t = 20$ seconds. Ans. 30 cubic inches per second.

4. The temperature of an iron rod increases along the rod in accordance with equation (1) of Art. 8. At a point 20 inches from the end of the rod ($x = 20$ inches), the value of T is 240° C. Find the value of k and state the unit in terms of which k is expressed. Ans. 0.6 degree per inch per inch.

5. Using the value of k from problem 4, find the temperature gradient (*a*) at the point where $x = 5$ inches and (*b*) at the point where $x = 20$ inches. Ans. (*a*) 6 degrees per inch, (*b*) 24 degrees per inch.

6. (*a*) Find the average temperature gradient along the iron rod of problem 4 between $x = 0$ and $x = 20$ inches. (*b*) Find the average temperature gradient between $x = 10$ inches and $x = 11$ inches. Ans. (*a*) 12 degrees per inch, (*b*) 12.6 degrees per inch.

9. Arbitrary variations.—One of the most important things in calculus is to consider how fast such an expression as x^2 or x^3 or $\sin x$ changes when x is arbitrarily assumed to grow steadily in value. Such variations may be called *arbitrary variations*. The very great importance of arbitrary variations will be understood when the student reaches Arts. 20 and 21.

10. A purely algebraic example of an arbitrary variation.—Having given the equation

$$y = ax^2 \tag{1}$$

let it be required to find the limiting value of the quotient $\frac{\Delta y}{\Delta x}$ when the arbitrary increment of x, namely Δx, approaches zero. Writing $y + \Delta y$ for y and writing $x + \Delta x$ for x in equation (1) we have

$$y + \Delta y = a(x + \Delta x)^2 \tag{2}$$

or

$$y + \Delta y = ax^2 + 2ax \cdot \Delta x + a(\Delta x)^2 \tag{3}$$

whence, subtracting equation (1) from equation (3) member by

member, we have

$$\Delta y = 2ax \cdot \Delta x + a(\Delta x)^2 \tag{4}$$

and dividing both members by Δx we have

$$\frac{\Delta y}{\Delta x} = 2ax + a \cdot \Delta x \tag{5}$$

But $a \cdot \Delta x$ approaches zero when Δx approaches zero, and therefore it is evident that the limiting value of $\frac{\Delta y}{\Delta x}$ is $2ax$. The limiting value of $\frac{\Delta y}{\Delta x}$ is always represented by $\frac{dy}{dx}$. Therefore we have:

$$\frac{dy}{dx} = 2ax \tag{6}$$

11. Two more purely algebraic examples of arbitrary variations. —(*a*) Having given the equation

$$y = ax^3 \tag{1}$$

let it be required to find the limiting value of $\frac{\Delta y}{\Delta x}$ when the arbitrary increment of x approaches zero. Writing $y + \Delta y$ for y and writing $x + \Delta x$ for x in equation (1) we have

$$y + \Delta y = a(x + \Delta x)^3 \tag{2}$$

or

$$y + \Delta y = ax^3 + 3ax^2 \cdot \Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3 \tag{3}$$

whence, subtracting equation (1) from equation (3) member by member, we have

$$\Delta y = 3ax^2 \cdot \Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3 \tag{4}$$

or

$$\frac{\Delta y}{\Delta x} = 3ax^2 + 3ax \cdot \Delta x + a(\Delta x)^2 \tag{5}$$

Now $3ax \cdot \Delta x$ and $a(\Delta x)^2$ both approach zero when Δx approaches zero, and therefore it is evident that the limiting value of $\frac{\Delta y}{\Delta x}$ is

$3ax^2$; that is,

$$\frac{dy}{dx} = 3ax^2 \tag{6}$$

(*b*) Having the equation

$$y = \frac{a}{x} \tag{7}$$

let it be required to find the limiting value of $\frac{\Delta y}{\Delta x}$ when the arbitrary increment of x approaches zero. Writing $y + \Delta y$ for y and writing $x + \Delta x$ for x in equation (7) we have

$$y + \Delta y = \frac{a}{x + \Delta x} \tag{8}$$

Subtracting equation (7) from equation (8) member by member, we have

$$\Delta y = \frac{a}{x + \Delta x} - \frac{a}{x} \tag{9}$$

Reducing the two fractions to a common denominator we have:

$$\Delta y = -\frac{a \cdot \Delta x}{x^2 + x \cdot \Delta x} \tag{10}$$

whence

$$\frac{\Delta y}{\Delta x} = -\frac{a}{x^2 + x \cdot \Delta x} \tag{11}$$

but $x^2 + x \cdot \Delta x$ approaches x^2 as its limit when Δx approaches zero, and therefore the limiting value of $\frac{\Delta y}{\Delta x}$ is $-\frac{a}{x^2}$; that is:

$$\frac{dy}{dx} = -\frac{a}{x^2} \tag{12}$$

The derivative $\frac{dy}{dx}$ is sometimes called the rate of change of y with respect to x.

PROBLEMS.

The following problems can be solved by the methods employed in Arts. 10 and 11. In each case find the rate of change of y with respect to x; a, b, c, m and n are constants.

1. $y = ax$, $\quad \frac{dy}{dx} = a.$

2. $y = ax^5$, $\quad \frac{dy}{dx} = 5ax^4.$

3. $y = mx^2 + n$, $\quad \frac{dy}{dx} = 2mx.$

4. $y = ax^2 + bx + c$, $\quad \frac{dy}{dx} = 2ax + b.$

5. $y = ax^3 + bx^2$, $\quad \frac{dy}{dx} = 3ax^2 + 2bx.$

6. $y = \frac{a}{x^2}$, $\quad \frac{dy}{dx} = -\frac{2a}{x^3}.$

7. $y = \frac{a}{x^3}$, $\quad \frac{dy}{dx} = -\frac{3a}{x^4}.$

8. $y = \frac{a}{bx^2 + x}$, $\quad \frac{dy}{dx} = \frac{-a(2bx + 1)}{(bx^2 + x)^2}.$

9. $y = \frac{3x}{x + 3}$, $\quad \frac{dy}{dx} = \frac{9}{(x + 3)^2}.$

10. $y = \frac{ax}{b - x}$, $\quad \frac{dy}{dx} = \frac{ab}{(b - x)^2}.$

11. $y = \frac{2x - 5}{x + 2}$, $\quad \frac{dy}{dx} = \frac{9}{(x + 2)^2}.$

12. $y = \frac{x}{(x - 1)^2}$, $\quad \frac{dy}{dx} = -\frac{x + 1}{(x - 1)^3}.$

12. Functions.—When a spring is subjected to a stretching force F, a certain elongation e is produced as shown in Fig. 2. *For*

every value of e *there is a definite corresponding value of* F. When two quantities are associated in this way either one is said to be a

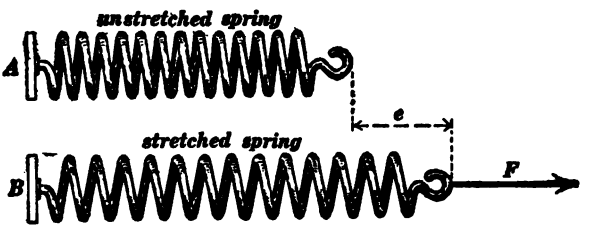

Fig. 2.

The elongation of the spring is a function of the stretching force.

function of the other. Thus the elongation e produced by a given force F in Fig. 2 is a function of F; or the force F required to produce a given elongation e is a function of e. The

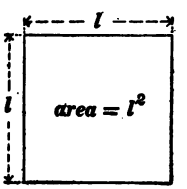

Fig. 3.

The area of a square is a function of length of side.

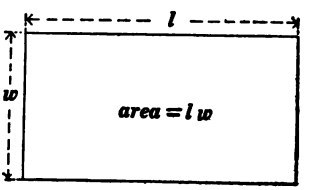

Fig. 4.

The area of a rectangle is a function of length and width.

meaning of the word function is further illustrated by Figs. 3, 4, 5, 6, 7 and 8.

13. Dependent and independent variables.—If l in Fig. 3 is assumed to be arbitrarily variable it is called an *independent variable*, and the area of the square is called a *dependent variable*, because its value is determined or fixed when the value of l is given. The quantity of air pumped into the cylinder in Fig. 7 may be as much as one pleases, the temperature of the cylinder

and air may be increased or decreased at will, and the volume of the cylinder may be changed by moving the piston. Therefore

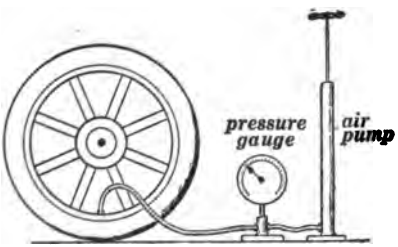

Fig. 5.

The pressure of the air is a function of the amount of air pumped into the tire.

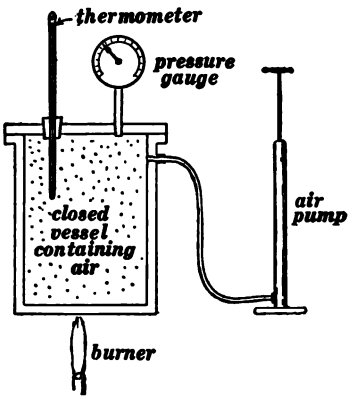

Fig. 6.

The pressure of the air in the vessel is a function of the amount of air pumped into the vessel and of the temperature.

the three quantities, (*a*) quantity of air, (*b*) temperature of the air, and (*c*) volume of space occupied by the air are called *independent variables*, and the pressure of the air is called a *dependent variable*,

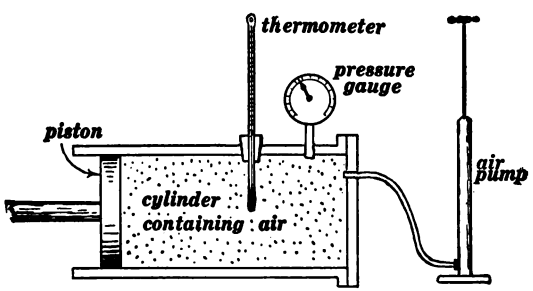

Fig. 7.

The pressure of the air in the cylinder is a function of (*a*) the amount of air pumped into the cylinder, (*b*) the temperature, and (*c*) the volume of the space occupied by the air.

because its value is fixed or determined when the quantities (a), (b) and (c) are given.

From this discussion it is evident that a dependent variable may be a function of one, two, three or more independent variables.

14. Natural functions and artificial functions.—We have heretofore taken the point of view that two quantities must be connected physically if either is to be a function of the other. Such a function may be called a *natural function*. It is, however, of very great importance to consider functional relations which grow out of or are expressed by algebraic equations. Such functions may be called *artificial functions*. Some idea of the importance of artificial functions may be obtained by looking back at Articles 4 and 6, and 7 and 8. Natural functions can be in many cases expressed algebraically; indeed no function can be handled in calculus unless the function is expressed algebraically.

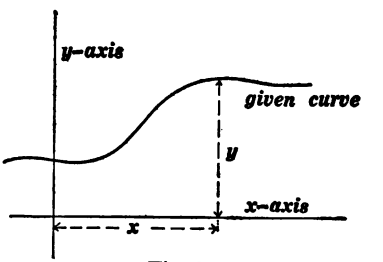

Fig. 8.

The ordinate of a point on a curve is a function of the abscissa of the point.

15. Tabulation of a function.—When one quantity is a function of another it is often convenient to express the functional relation by means of a table giving pairs of corresponding values of the two quantities. An ordinary table of logarithms is such a table, a table of sines or cosines is such a table, the mortality table* used in life insurance is such a table.

When a functional relation is determined by experiment the

* If one were to consider all men in the United States who are now forty years of age and keep a future record of their deaths one would find that their average age at death would be, say, sixty-five years. This is expressed by saying that the expectancy of life at forty years is twenty-five years, because on the average a man forty years old may expect to live twenty-five years. The expectancy of life is, of course, less and less with increasing age, and there is a definite expectancy of life corresponding to each age. That is, expectancy of life is a function of age.

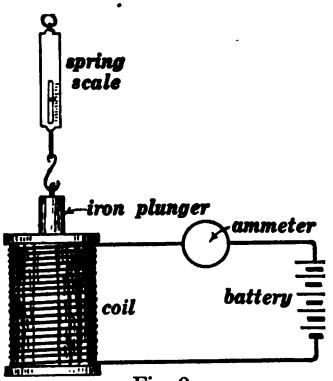

Fig. 9.

observed quantities are always arranged in tabular form. For example Fig. 9 represents an ammeter arranged to measure the electric current flowing through a coil of wire, and a spring scale arranged to measure the force with which an iron plunger is pulled into the coil; and the accompanying table shows a series of observed values of current in amperes and the corresponding pulls in pounds.

OBSERVED RELATION BETWEEN CURRENT AND PULL IN FIG. 9.

Current in Amperes.	Pull on Plunger in Pounds.
1.0	*3.9*
2.0	*23.5*
3.0	*40.0*
4.0	*48.1*

16. Graphical representation of a function.—The relation between the pull of the spring and the reading of the ammeter in Fig. 9, as shown in the table of Art. 15, may be represented graphically by a curve of which the abscissas represent ammeter readings and of which the ordinates represent corresponding readings of the spring scale. Such a curve is shown in Fig. 10. In the same way an algebraic function may be represented graphically. Thus the curve in Fig. 11 represents the function $y = x^3$.

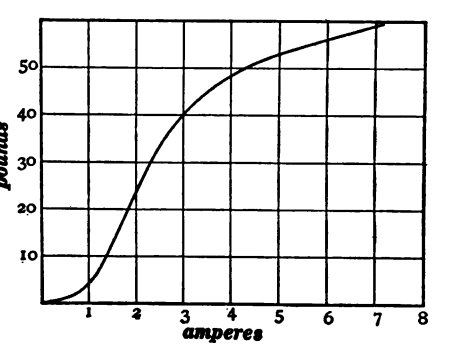

Fig. 10.

17. Derivative of a function.—Let y be a function of x and let Δy be the increment of y due to an arbitrary increment of x. Then the limiting value of the quotient $\frac{\Delta y}{\Delta x}$ as Δx approaches zero is called *the derivative of y with respect to x* or *the rate of change of y with respect to x.**

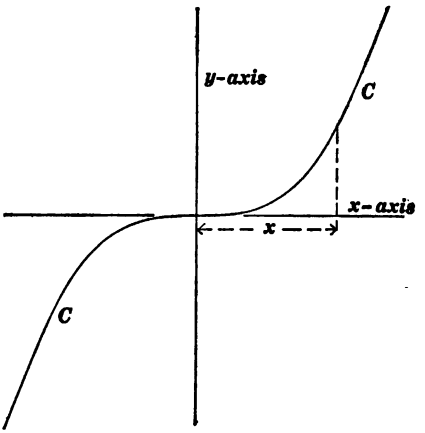

Fig. 11.

When one wishes to speak in general terms of any function of x and its derivative, the function may be represented by $f(x)$ or by $\phi(x)$ and the derivative of the function with respect to x may be represented by $f'(x)$ or by $\phi'(x)$.

Meaning of a derivative.—Let y be a function of x and let x be assumed to grow steadily at a definite rate, *then y grows $\frac{dy}{dx}$ times as fast as x at each instant.* For example, let $y = x^2$; then $\frac{dy}{dx} = 2x$, according to Art. 10, and the following relations exist:

y increases $2x(=20)$ times as fast as x while x is passing through the value $x = 10$,
y increases $2x(=22)$ times as fast as x while x is passing through the value $x = 11$,
y increases $2x(=24)$ times as fast as x while x is passing through the value $x = 12$,
etc., etc., etc.

As another example, let $y = \frac{1}{x}$; then $\frac{dy}{dx} = -\frac{1}{x^2}$, according to Art. 11, and the following relations exist:

* Sometimes also called *the differential coefficient of y with respect to x.*

y decreases $\frac{1}{x^2}\left(=\frac{1}{4}\right)$ as fast as x *increases* when x is passing through the value 2,

y decreases $\frac{1}{x^2}\left(=\frac{1}{9}\right)$ as fast as x *increases* when x is passing through the value 3,

etc., etc., etc.

A negative value of $\frac{dy}{dx}$ shows that y decreases as x increases.

The derivative of a function of x is itself a function of x. —If $y = ax^2$, then $\frac{dy}{dx} = 2ax$, according to Art. 10. In this case it is evident that $\frac{dy}{dx}$ is a function of x because it is equal to $2ax$ and it has a definite value for every value of x. If $f(x)$ represents any function of x, then its derivative $f'(x)$ is in general a function of x also.

18. Graphical representation of derivative. Slope of a curve at a point.—Consider the curve CC, Fig. 12. This curve represents

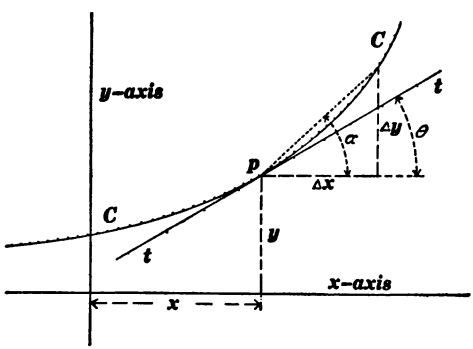

Fig. 12.

a definite function, that is, the ordinate y is a definite function of the abscissa x, the point p being anywhere on the curve. If x is increased by the amount Δx, then y will be increased by the

amount Δy as shown, and the ratio $\frac{\Delta y}{\Delta x}$ is equal to the tangent of the angle α.

Draw the line tt touching the curve at p as shown. As Δx approaches zero the angle α becomes more and more nearly equal to the angle θ, in fact θ is the limiting value of α, and, of course, the limiting value of $\frac{\Delta y}{\Delta x}$ is $\frac{dy}{dx}$. Therefore, since $\tan \alpha = \frac{\Delta y}{\Delta x}$, we must have $\tan \theta = \frac{dy}{dx}$. That is, *the function y being represented by the ordinates of a curve, the derivative $\frac{dy}{dx}$ is represented by the steepness or grade of the curve at each point.*

PROBLEMS.

1. A pail is 10 inches in diameter, so that the volume of water in the pail is $y = \frac{100\pi}{4} x$, where x is the depth of the water in the pail. Find how fast water must be poured into the pail to cause the water level to rise at a velocity of 4 inches per second. Ans. 100π cubic inches per second.

2. Water flows at a constant rate of 30 cubic inches per second into a metal cone of which the dimensions are as shown in Fig. *p*2. Find the velocity at which the water level rises (*a*) when $x = 5$ inches, and (*b*) when $x = 15$ inches. Ans. (*a*) 2.715 inches per second, (*b*) 0.302 inch per second.

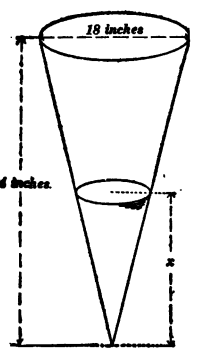

Fig. *p*2.

3. The sides of a square are growing at the rate of 5 inches per second. Find the rate of growth of the area of the square (*a*) when the sides of the square are 10 inches and (*b*) when the sides of the square are 20 inches. Ans. (*a*) 100 square inches per second, (*b*) 200 square inches per second.

4. The radius of a circle is growing at the rate of 5 inches per second. Find the rate of growth of the area of the circle (*a*)

when the radius is 10 inches and (*b*) when the radius is 20 inches. Ans. (*a*) 100π square inches per second, (*b*) 200π square inches per second.

5. The edges of a cube are growing at the rate of 5 inches per second. Find the rate of growth of the volume of the cube when the edges are 10 inches long. Ans. 1,500 cubic inches per second.

6. Air is blown into a soap bubble at the rate of 10 cubic inches per second. Find the rate of increase of the radius of the bubble when the radius has the values (*a*) 1 inch and (*b*) 4 inches. Ans. (*a*) 0.796 inch per second, (*b*) 0.050 inch per second.

7. A man 6 feet tall walks at a speed of 4 miles per hour under a lamp which is 10 feet from the ground. Find how fast the tip of the man's shadow travels when the horizontal distance from the lamp to the man is 20 feet. Ans. 10 miles per hour.

8. The two sides of a right angled triangle are 40 inches and x inches, and the hypothenuse is y inches. The side x is growing at the rate of 5 inches per second. How fast is y growing when $x = 30$ inches? Ans. 3 inches per second.

9. The two sides of a right angled triangle are x and y, and the hypothenuse is 50 inches. The side x is growing at the rate of 4 inches per second. How fast is y decreasing (*a*) when $x = 30$ inches, and (*b*) when $x = 40$ inches. Ans. (*a*) 3 inches per second. (*b*) 5.33 inches per second.

10. The observed temperatures of a vessel of cooling water after 1 minute, after 2 minutes, and so forth are as follows:

Elapsed Time in Minutes, t.	Observed Temperatures, T.
0	92.0
1	85.3
2	79.5
3	74.5
5	67.0
7	60.5
10	53.5
15	45.0
20	39.5

Plot these values of t and T, using the best grade of squared paper, draw a smooth curve through the plotted points, draw tangents to the curve at the points corresponding to (*a*) $T = 80°$, (*b*) $T = 65°$, and (*c*) $T = 50°$, and determine the rate of decrease of the temperature at each point by measuring the intercepts of these tangents on the axes of reference.

11. Find for what values of x the function $y = 6x^2 - 12x$ is an increasing function and for what values of x it is a decreasing function. Ans. y is an increasing function when x is greater than 1, and a decreasing function when x is less than 1.

12. Plot the values of x and y, where $y = x^3$, using the best grade of squared paper, draw a smooth curve through the plotted points, draw tangents to the curve at the points corresponding to $x = 1$, $x = 2$, and $x = 3$, and determine the corresponding values of $\frac{dy}{dx}$ by measuring the intercepts of the tangents on the axes of reference. Compare the values of $\frac{dy}{dx}$ thus found with the values as calculated from the formula $\frac{dy}{dx} = 3x^2$.

13. Plot the values of x and y where $y = \log_{10} x$. Take the values of $\log_{10} x$ from an ordinary table of logarithms and use the following values of x: 2, 3, 4, 5, 6, 7, 8, 9 and 10. Draw a smooth curve through the plotted points, draw tangents to the curve at the points corresponding to $x = 4$, $x = 6$ and $x = 8$, and determine the corresponding values of $\frac{dy}{dx}$ by measuring the intercepts of these tangents on the axes of reference. Compare the values of $\frac{dy}{dx}$ thus found with the values as calculated by the formula $\frac{dy}{dx} = \frac{\log_{10} e}{x} = \frac{0.4343}{x}$.

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19. Functions which have the same derivative.—The curve A, Fig. 13, defines y' as a function of x, and the curve B defines y

as a function of x. The steepness or grade of curve A is everywhere the same as the steepness or grade of curve B, that is, the derivative $\frac{dy'}{dx}$ is equal to the derivative $\frac{dy}{dx}$ for each value of x, or expressed as an equation we have

$$\frac{dy'}{dx} = \frac{dy}{dx} \tag{1}$$

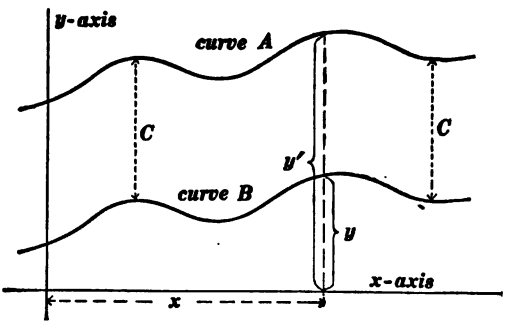

Fig. 13.

From the figure it is evident that the difference $y' - y$ is everywhere equal to the constant quantity C. Consequently when equation (1) is true we have

$$y' - y = \text{a constant} \tag{2}$$

That is to say, *two functions whose derivatives are equal have a constant difference.*

Examples.—When two men save money at the same rate there is a constant difference between the amounts they have saved; or, if they start even, their savings are equal. When two trains travel at the same speed along a straight track they remain at a constant distance apart. The two functions ax^2 and $ax^2 + b$ have the same derivative with respect to x, b being a constant.

20. Example showing the use of calculus. Work required to stretch a spring.—Let W be the work required to stretch a

spring from condition A to condition B, Fig. 2. It is evident that W is a function of e, and it is desired to find an algebraic expression for this function; that is, it is desired to find an equation expressing W in terms of e. *To do this the first step is to find an expression for the derivative* $\frac{dW}{de}$ by considering the amount of work ΔW that must be done to produce a slight additional elongation Δe.

The force F in Fig. 2 is, according to Hooke's law, proportional to e; that is,

$$F = ke \tag{1}$$

where k is a constant.

Imagine the force F to be increased by the amount ΔF so that the elongation would be increased by the amount Δe. Then the work done would be *greater than* $F \cdot \Delta e$ and *less than* $(F + \Delta F) \cdot \Delta e$ because the *average value* of the force which acts on the spring while the added elongation Δe is being produced is greater than F and less than $F + \Delta F$. Therefore

$$\Delta W \text{ is greater than } F \cdot \Delta e \tag{2}$$

and

$$\Delta W \text{ is less than } (F + \Delta F) \cdot \Delta e \tag{3}$$

whence, dividing by Δe, we have:

$$\frac{\Delta W}{\Delta e} \text{ is greater than } F \text{ and less than } F + \Delta F \tag{4}$$

Therefore $\frac{\Delta W}{\Delta e}$ approaches F as its limit as ΔF (and also Δe) approaches zero. That is,

$$\frac{dW}{de} = F \tag{5}$$

or, using the value of F from equation (1), we have

$$\frac{dW}{de} = ke \tag{6}$$

Now in Art. 10 it is shown that ax^2 is a function whose derivative is $2ax$. Therefore:

ae^2 is a function whose derivative is $2ae$ (7)

Therefore, substituting $k = 2a$ (or $a = \frac{1}{2}k$) in (7) we have:

$\frac{1}{2}ke^2$ is a function whose derivative is ke (8)

The work W is also a function whose derivative is ke, according to equation (6). Therefore, according to Art. 19, we must have

$$W = \tfrac{1}{2}ke^2 + C \tag{9}$$

where C is a constant. To determine the value of the constant C we must *know* the value of W for some particular value of e, and of course we know that $W = 0$ when $e = 0$, that is, no work is required when the spring is not stretched at all. Therefore, substituting this pair of values of W and e in equation (9), we have

$$0 = C \tag{10}$$

Substituting this value of C in equation (9) we have

$$W = \tfrac{1}{2}ke^2 \tag{11}$$

which is the desired expression for the work W.

21. Another example showing the use of calculus. The area under a parabola.—The curve cc, Fig. 14a, represents a parabola of which the equation is:

$$y = px^2 \tag{1}$$

and it is required to find an expression for the area A.

Imagine the value of x in Fig. 14a to increase by the amount Δx, then the increment of A is the area of the shaded strip in Fig. 14b. The width of this strip is Δx and its average height is greater than y and less than $y + \Delta y$. Therefore, writing ΔA for the area of the narrow strip, we have:

ΔA is greater than $y \cdot \Delta x$ (2)

and

ΔA is less than $(y + \Delta y) \cdot \Delta x$ (3)

ce, dividing by Δx, we have:

$$\frac{\Delta A}{\Delta x} \text{ is greater than } y \text{ and less than } y + \Delta y \tag{4}$$

Therefore $\frac{\Delta A}{\Delta x}$ approaches y as a limit when Δx and Δy approach zero. But in this discussion y stands for px^2, according to equation (1); therefore we have:

$$\frac{dA}{dx} = px^2 \tag{5}$$

In Art. 11 it was shown that:

$$ax^3 \text{ is a function whose derivative is } 3ax^2 \tag{6}$$

Therefore substituting $\frac{p}{3}$ for a we have:

$$\tfrac{1}{3}px^3 \text{ is a function whose derivative is } px^2 \tag{7}$$

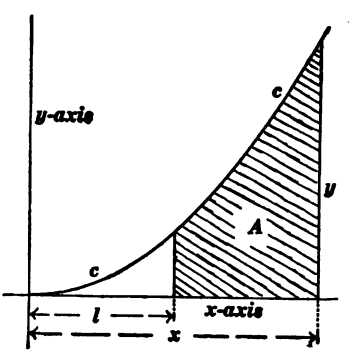

Fig. 14a.

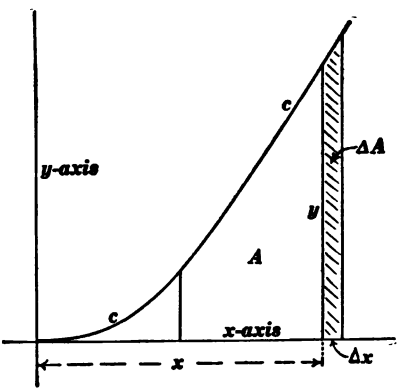

Fig. 14b.

The area A is also a function whose derivative with respect to x is px^2 according to equation (5), and therefore, according to Art. 19, we must have:

$$A = \tfrac{1}{3}px^3 + C \tag{8}$$

where C is a constant. To determine the value of the constant C we must *know* the value of A for some particular value of x.

An inspection of Fig. 14*a* shows that $A = 0$ when $x = l$. Therefore, substituting this pair of values of A and x in equation (8) we have:

$$0 = \tfrac{1}{3}pl^3 + C \tag{9}$$

so that

$$C = -\tfrac{1}{3}pl^3 \tag{10}$$

Substituting this value of C in equation (8) we have:

$$A = \tfrac{1}{3}px^3 - \tfrac{1}{3}pl^3 \tag{11}$$

which is the desired expression for the area A in Fig. 14*a*.

It is to be noted that x is a variable in the above discussion because we have arbitrarily made it vary in the derivation of equation (5), whereas p and l are both constant.

PROBLEMS.

Using the method of Arts. 20 and 21 set up the following derivatives:

1. $\dfrac{dV}{dx} = \tfrac{1}{3}\pi x^2$, where V is the volume of a cone of which the half-angle at the apex is 30° and of which the altitude is x.

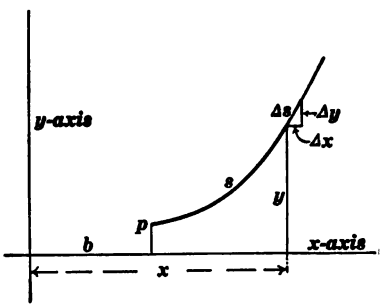

Fig. *p*4.

2. $\dfrac{dA}{dx} = \tfrac{4}{3}\pi x$, where A is the area of the curved surface of the cone of problem 1.

3. $\dfrac{dV}{dx} = \dfrac{\sqrt{3}}{8}\pi x^2$, where V is the volume of the cone of problem 1 and x is its slant height.

4. The length s of the heavy portion (starting at the point p where $x = b$) of the given curve in Fig. *p*4 is a function of x. Find an expression for the derivative $\dfrac{ds}{dx}$. Ans. $\dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$.

Note. The chord which corresponds to the arc Δs in Fig. *p*4 is equal to $\sqrt{(\Delta x)^2 + (\Delta y)^2}$. Therefore we have:

$$\frac{\text{arc}}{\text{chord}} = \frac{\Delta s}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{\Delta s}{\Delta x} \cdot \frac{1}{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}}$$

But for any smooth curve the limit of $\frac{\text{arc}}{\text{chord}}$ is unity when arc and chord approach zero. The recognition of this limit is axiomatic, and it forms the basis of the idea of length of a curve.

5. Consider the parabola in Fig. 14*a* of which the equation is $y = px^2$. Let s be the length of the portion of the parabolic arc between $x = l$ and $x = x$ (meaning any value of x whatever). Find an expression for the derivative $\frac{ds}{dx}$. Ans. $\frac{ds}{dx} = \sqrt{1 + 4p^2x^2}$.

6. The shaded area A in Fig. *p*6 is a function of the angle θ. Find an expression for the derivative $\frac{dA}{d\theta}$, the angle θ being expressed in radians and the equation to the curve *cc* being $r = k\theta + b$, where k and b are constants. Ans. $\frac{dA}{d\theta} = \frac{1}{2}(k\theta + b)^2$.

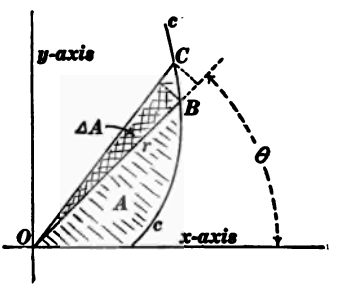

Fig. *p*6.

Note. The increment ΔA which corresponds to a given increment $\Delta\theta$ is shown in the figure. The increment ΔA is greater than the area of the circular sector OB, and less than the area of the circular sector OC.

7. A very slim stick l centimeters long rotates like the spoke of a wheel about an axis which passes through one end of the stick. The speed of the stick is n revolutions per second and the mass of the stick is m grams per centimeter. The kinetic energy, W, of the stick is a function of l. Find an expression for the derivative $\frac{dW}{dl}$. Ans. $\frac{dW}{dl} = 2\pi^2n^2ml^2$.

Note. The added mass due to an increment of l is $m \cdot \Delta l$. The velocity of this added mass ranges from $2\pi nl$ to $2\pi n(l + \Delta l)$, and therefore the kinetic energy of the added mass is greater than $\frac{1}{2}m \cdot \Delta l \cdot (2\pi nl)^2$ and less than $\frac{1}{2}m \cdot \Delta l \cdot [2\pi n(l + \Delta l)]^2$. Energy is here expressed in ergs.

8. Let W be the kinetic energy in ergs of a steel disk 10 centimeters thick and r centimeters in radius rotating like a buzz saw at a speed of n revolutions per second. Then W is a function of r. Find an expression for the derivative $\frac{dW}{dr}$.

Ans. $\frac{dW}{dr} = 312\pi^3 n^2 r^3$.

Note. The density of steel is 7.8 grams per cubic centimeter.

9. Let W be the work done in pushing the piston from the position A to the position B in Fig. $p9$, the area of the piston

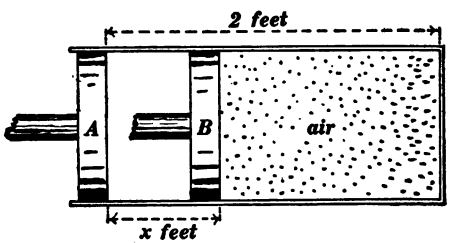

Fig. $p9$.

being a square inches, the initial pressure of the air when the piston is at A being 15 pounds per square inch, and the air being kept at unchanging temperature during the compression so that the increasing pressure is inversely proportional to the decreasing volume. Find an expression for the derivative $\frac{dW}{dx}$, x being expressed in feet and W in foot-pounds. Ans. $\frac{dW}{dx} = \frac{30a}{2 - x}$.

Note. The pressure of the air in the cylinder corresponding to any given value of x is $\frac{2}{2 - x}$ times 15 pounds per square inch. The pressure of the

outside air is to be ignored, that is to say, the outside air is not supposed to help push the piston.

10. A cylinder 100 centimeters high and 50 square centimeters section is filled with liquid. The density of the liquid increases uniformly from one gram per cubic centimeter at the top to two grams per cubic centimeter at the bottom of the vessel. Let M be the mass in grams of the liquid between the top of the vessel and a point x centimeters from the top. Find an expression for the derivative $\frac{dM}{dx}$. Ans. $\frac{dM}{dx} = 50 + \frac{1}{2}x$.

---

22. Derivative notation and differential notation.—The central idea of differential calculus is *the limiting value of* $\frac{\Delta y}{\Delta x}$ *when both* Δx *and* Δy *approach zero,* y being a function of x; and it is desirable, especially in an elementary treatise on calculus, to use a system of notation and nomenclature which will keep this fundamental idea constantly before the reader. It is for this reason that the complicated group of letters $\frac{dy}{dx}$ is used for the derivative, it reminds one of the ratio $\frac{\Delta y}{\Delta x}$, and the use of d instead of Δ is intended to bring to mind the limiting value of $\frac{\Delta y}{\Delta x}$.

Let us consider a simple function; for example let:

$$y = ax^2 \tag{1}$$

of which the derivative is:

$$\frac{dy}{dx} = 2ax \tag{2}$$

In this equation, as heretofore, $\frac{dy}{dx}$ is thought of as a single symbol, the meaning of which is explained in Art. 17; and this

equation exemplifies the so-called *derivative notation*, a notation in which the symbol $\frac{dx}{dy}$ is never separated into parts.

If, however, we look upon dx and dy as being separate and distinct symbols of quantity, then, according to the simple rules of algebra, it should be allowable to write equation (2) thus:

$$dy = 2ax \cdot dx \tag{3}$$

This equation exemplifies what is called the *differential notation*, a notation in which dy and dx are treated as separate symbols of quantity; dy is called the *differential of* y, and dx is called the *differential of* x. The differential notation is used almost exclusively in integral calculus (see Art. 26), and it is nearly always used by writers on applied mathematics.

23. Rigorous definition of dy and dx as quantities.—After a body falls for one second it has a velocity of 32 feet per second, *which means that the body would travel* 32 *feet during the next second if its velocity were to remain unchanged.* Similarly, if $y = x^2$, then y changes $2x$ times as fast as x, that is, y changes twenty times as fast as x at the instant that x passes the value $x = 10$, *which means that* dy (**the actual finite change of** y) *would be twenty times as large as* dx (**the actual finite change of** x) *if* y *were to continue to change twenty times as fast as* x. This is the exact meaning of the equation $dy = 2x \cdot dx$, and on the basis of this interpretation dy and dx are ordinary finite quantities. Whenever the question comes up as to the exact meaning of a differential, the above statement must be called to mind.

It is entirely out of the question, however, to use the words *would be* and *if* every time we speak of a differential, and it is undesirable even to think in such conditional terms; as a matter of fact nearly all writers on applied mathematics use the word *differential* in a composite sense. Instead of using the expressions Δy and Δx to designate actual increments of y and x in the early stages of a discussion and changing to the expressions

dy and dx when the limiting relation is found, the letter d is used first and last. The advantage of this procedure is that it favors simplicity and directness of speech, and it permits a degree of concreteness which would otherwise be unattainable.* It is a great advantage for the clear understanding of an abstract discussion if the utmost possibility of concreteness be made use of.† The above-mentioned procedure does, however, alway carry with it a weakened emphasis upon the essential steps in the argument which leads one to the limiting value of $\frac{\Delta y}{\Delta x}$.

Another point of view. The equation $dy = 2x \cdot dx$ does represent a point of view concerning the limiting relation between Δy and Δx which is somewhat different from the point of view which is taken in Arts. 6, 8, 10, 11, 20 and 21; and this point of view is set forth in the following discussion:

(*a*) *The terms large and small are purely relative.* When we speak of a quantity as being large or small we always have in mind more or less distinctly another quantity of the same kind as a basis of comparison. Thus six miles is small as a distance to travel by rail, but six miles would be enormously large as a distance to jump. A river six miles long would be very short, but a six-mile nose would be absurdly long! Neither largeness nor smallness has any meaning as applied to a single quantity. One cannot even say, for example, that a large distance is a distance which is expressed as a large number of units, because the unit itself may be as large as we please; thus the distance around the earth is about 0.000,000,003 of a light-year, the light-year being a unit of distance which is sometimes used by astronomers. Nor can one say that a small distance is a distance which is expressed as a small fraction of a unit, because the unit itself may be as small as we please; thus the thickness of this paper is about 100 microns, the micron being a familiar unit of length to the microscopist. It is entirely possible to have two quantities both as small as you please, and yet one of the quantities may be as large as

* For example, the verbal part of the discussion on pages 115–119 would become greatly involved if one did not frankly speak of a differential as a vanishing increment, and the complicated argument in the note to problem 15 on page 233 would become hopelessly involved!

† This is substantially the meaning of Helmholtz when he said "Es hat grosse Vortheile für das sichere Verständniss aller Abstractionen wenn man möglichst concrete Bilder davon zu machen sucht, selbst wenn diese manche Voraussetzung hinein bringen die für das Wesen der Sache nicht gerade nothwendig ist."

you please as compared with the other. Consider, for example, the chord c and the versed sine h as shown in Fig. 15. As the chord is moved slowly towards the left, h and c both approach zero and the ratio $\frac{c}{h}$ becomes larger and larger without limit; that is, c may be a million, or a million-million, or a million-million-million, or any number of times as large as h; and, although both h and c may be very small as compared with an inch or a foot, they are very far indeed from being approximately equal to each other.

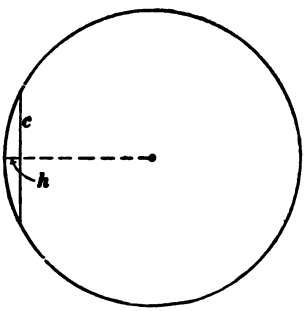

Fig. 15.

When the chord is moved slowly slowly to the left it grows shorter and the versed sine h also grows shorter.

(*b*) *Exact equality and approximate equality.* Let a and b be two quantities which are exactly equal to each other; then $a - b$ is zero and $\frac{a}{b}$ is unity. But let us consider what it means to say that a is *nearly* equal to b; it certainly does not mean that the difference $a - b$ is small. A man who owes \$2 and pays \$1 falls far short of his obligation, but a man who owes \$1000 and pays \$999 very nearly meets his obligation. That is to say, \$999 is very nearly equal to \$1000, whereas \$1 is not nearly equal to \$2, and yet the difference is \$1 in each case. It is impossible to judge as to the approximate equality of a and b by considering only the value of $a - b$. *Two quantities a and b are nearly equal when the ratio $\frac{a}{b}$ is nearly unity, and approximate equality approaches exact equality as the ratio $\frac{a}{b}$ approaches unity.*

(*c*) *Derivation of the expression $dy = 2x \cdot dx$.* Consider the simple function $y = x^2$, from which we get:

$$\Delta y = 2x \cdot \Delta x + (\Delta x)^2 \tag{1}$$

This equation may be written:

$$\frac{\Delta y}{2x \cdot \Delta x} = 1 + \frac{\Delta x}{2x} \tag{2}$$

from which it is evident that the ratio $\frac{\Delta y}{2x \cdot \Delta x}$ approaches unity as a limit when Δx approaches zero. That is to say, Δy and $2x \cdot \Delta x$ become as nearly equal as you please as Δy and Δx grow smaller and smaller, and the equation

$dy = 2x \cdot dx$ is intended to express this limiting relation of equality as here explained. That is to say, $dy = 2x \cdot dx$ is intended to mean exactly the same thing as $\frac{dy}{dx} = 2x$. To say that Δy and $2x \cdot \Delta x$ approach a limiting relation of equality is equivalent to the statement that *their ratio approaches unity as a limit.*

The awkwardness of the point of view here set forth as to the meaning of the equation $dy = 2x \cdot dx$ is that it places an unusual burden of meaning upon the sign of equality, whereas in all strictness this sign should never be used in algebra except to express the fact that two actual finite numbers have the same value. A careful consideration of this entire article and of article 26 will convince the student that the most serious difficulty of calculus is the awkwardness of the notation.

24. Rule for handling differentials.—As stated in Art. 23, writers on applied mathematics do not use the notation Δy, Δx, etc., in the earlier stages of a discussion where actual finite increments are being considered and then change to the notation dy, dx, etc., when the limiting relation is established; the usual procedure is to use the notation dy, dx, etc., both first and last. When this is done the differential of y (dy) and the differential of x (dx) are thought of in the early stages of a discussion as vanishing increments.

Consider the simple function $y = ax^3$. Then, according to Art. 11, we have:

$$\Delta y = 3ax^2 \cdot \Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3 \qquad (1)$$

and from Art. 11 we know that $\frac{dy}{dx} = 3ax^2$, or, using the differential notation, we have:

$$dy = 3ax^2 \cdot dx \qquad (2)$$

Therefore, by comparing equations (1) and (2), we see that the limiting relation between Δy and Δx is obtained by dropping terms containing squares and higher powers of Δx.

A vanishing increment is sometimes called an *infinitesimal;* and any given vanishing increment, dx for example, may be spoken of as an *infinitesimal of the first order.* Then $(dx)^2$ is

called an infinitesimal of the *second order*, $(dx)^3$ is called an infinitesimal of the *third order*, and so on.*

In any expression involving infinitesimals, only the terms of the lowest order have a bearing upon the LIMITING RELATION, which is always the thing to be determined; all terms containing higher order infinitesimals may be dropped. This rule is illustrated by making a comparison between equations (1) and (2) above; no attempt is here made to establish the rule rigorously. Indeed the rule does not have enough definite arithmetical meaning to admit of its being rigorously established as a general proposition, but it is extensively used nevertheless by writers on applied mathematics, and it is used in some of the later chapters of this book. See, for example, the note to problem 15 on page 233.

Another example.—Let it be required to find the limiting relation between Δy and Δx when $y^2 = ax^3$. Writing $(x + \Delta x)$ for x, and writing $(y + \Delta y)$ for y, we have:

$$y^2 + 2y \cdot \Delta y + (\Delta y)^2 = ax^3 + 3ax^2 \cdot \Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3 \quad (3)$$

whence, subtracting $y^2 = ax^3$, member from member, we have:

$$2y \cdot \Delta y + (\Delta y)^2 = 3ax^2 \cdot \Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3 \quad (4)$$

from which the desired limiting relation is found by dropping second and third order infinitesimals according to the above rule. We thus obtain:

$$2y \cdot dy = 3ax^2 \cdot dx \quad (5)$$

where d has been written for Δ to indicate the limiting relation. If it is desired to get the value of $\frac{dy}{dx}$ in terms of x alone the value $y = \sqrt{ax^3}$ can be substituted for y in equation (5).

25. The differential equation. Law of growth of a function.—Let y be an undetermined function of x concerning which it is

* If dy and dx have a finite ratio they are infinitesimals of the same order (first order, let us say), then the product $dy \cdot dx$ is an infinitesimal of the second order.

known that y increases $2ax$ times as fast as x. This property of the unknown function may be expressed by the equation:

$$\frac{dy}{dx} = 2ax \tag{1}$$

This equation expresses the law of growth of the function y; the independent variable x being assumed to grow steadily in value. Any equation which thus expresses the law of growth of a function is called a *differential equation.*

Examples.—(*a*) A man A saves money at a constant rate.* This statement is a verbal expression of a differential equation. To reduce the statement to algebra, let y be the amount of money saved by A, then the above statement is equivalent to

$$\frac{dy}{dt} = k \quad \text{or} \quad dy = k \cdot dt \tag{2}$$

where k is a constant.

(*b*) Every equation which involves a derivative or differentials is a differential equation.

26. Differentiation and integration.—The finding of the derivative of a function as exemplified in Arts. 10 and 11 is called *differentiation.*

Let y be an unknown function of x which is known to increase $2ax$ times as fast as x. That is to say, it is known concerning y that

$$\frac{dy}{dx} = 2ax \tag{1}$$

Now when one has previously found that ax^2 *is a function of* x *whose derivative is equal to* $2ax$ *one is able to recognize that the unknown function* y *whose law of growth is given by equation* (1) *must be equal to* $ax^2 + C$, *where* C *is a constant,* as explained in Art. 19. This recognition of a function from its derivative is called *integration.*

* Money saved is here assumed to be a continuous variable. See Art. 5.

Symbol of integration.—Let y stand for $ax^2 + C$, then

$$\frac{dy}{dx} = 2ax \tag{2a}$$

that is to say,

the derivative of $(ax^2 + C)$ *is equal to* $2ax$ (2b)

Or, using the differential notation which is explained in Art. 22, we have

$$dy = 2ax \cdot dx \tag{2c}$$

that is to say,

the differential of $(ax^2 + C)$ *is equal to* $2ax \cdot dx$ (2d)

All of these equations (2a), (2b), (2c) and (2d) express the same thing or the same relation, and this relation may be expressed in an inverse manner by saying:

$(ax^2 + C)$ *is the integral of* $2ax \cdot dx$ (3a)

or, in symbolic notation:

$$ax^2 + C = \int 2ax \cdot dx \tag{3b}$$

The symbol $\int$ means *integral of.* The undetermined constant C is called the *constant of integration.*

A differential equation does not completely determine the function whose law of growth it expresses.—This is evident when we consider that the function $y + C$ has the same derivative (the same law of growth) whatever the value of the constant C may be.

Thus the amount of a man's savings cannot be calculated when his rate of saving *alone* is known; one must know also how much money the man had on a given date. Thus a man who has \$1,000 on January 1, 1912, and who saves \$10 per month would have $1{,}000 + 10m$ dollars at any time m months after January 1. The amount of a man's savings is completely determined as a function of elapsed time by knowing (*a*) his rate of saving and (*b*) how much he has on a given date.

In general a function y is completely determined* by knowing (a) its law of growth and (b) its value corresponding to any given value of the independent variable x or t as the case may be. This matter is fully illustrated in Arts. 20 and 21.

Indefinite integrals.—Let y be a function concerning which nothing is known except that $\frac{dy}{dx}$ is equal, say, to $2ax$, or, using differential notation, we know that

$$dy = 2ax \cdot dx \tag{1}$$

Then, according to Arts. 10 and 19, we know that y must be given by the equation:

$$y = ax^2 + C \tag{2}$$

where C is a constant which may have any value whatever. Equation (2) expresses what is called the *indefinite integral* of equation (1).

Definite integrals.—Let y be a function concerning which nothing is known except that $\frac{dy}{dx} = 2ax$, or, using differential notation, we know that

$$dy = 2ax \cdot dx \tag{3}$$

Let it be required to find how much y grows in value while x grows from any given value $x = l$ to any other given value $x = L$. Now, according to equation (2), we have $y = al^2 + C$ when $x = l$, and $y = aL^2 + C$ when $x = L$. Therefore we may find the desired *growth* in value of y, by subtracting the first value of y ($= al^2 + C$) from the second value of y ($= aL^2 + C$), which gives $aL^2 - al^2$. This growth of y from $x = l$ to $x = L$ is called the **definite integral** of equation (3) between $x = l$ and $x = L$. The two given values of x, namely l and L, are called the **boundaries** or **limits**† of the definite integral.

* This statement will have to be modified when we come to consider differential equations of higher orders, and when we come to consider partial differential equations.

† The word *limit* as here used must not be confused with the word *limit* as used in Arts. 9 and 10.

Examples of definite integrals.—The work W required to stretch a spring from zero elongation to any given elongation e is a function of e whose derivative with respect to e is ke, as explained in Art. 20, that is, W increases ke times as fast as e; and the total increase or growth of W while e is increasing from zero to any given value e is the desired value of W corresponding to the given value of e.

The area A in Fig. 14 is a function of x whose derivative with respect to x is px^2, as explained in Art. 21, that is, A increases px^2 times as fast as x; and the total increase or growth of A while x is increasing from l to any given value x is the desired value of A corresponding to the given value of x.

A derivative must have a finite value everywhere between and at the limits of a definite integral, otherwise the value of the integral cannot be found as above explained. Thus if a man saves money

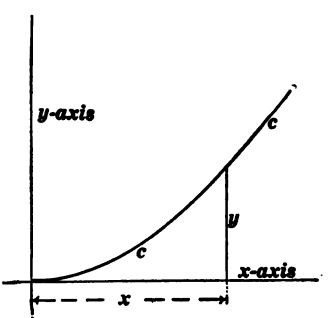

Fig. 16*a*.

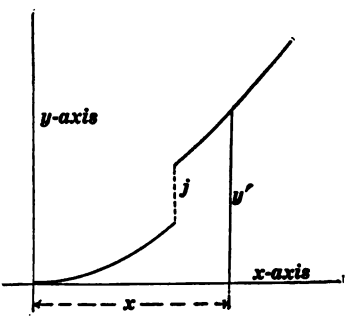

Fig. 16*b*.

continuously at a known rate it is easy to calculate the amount of money saved in a given time by multiplying the known rate by the time; but what if the man should find a thousand-dollar bill during the given time? The finding of a thousand-dollar bill is a thing which cannot be expressed as a rate, unless indeed we choose to speak of it as an infinite rate which is meaningless from the physical point of view and meaningless from the arithmetical point of view.

Figure 16*a* shows a curve of which the slope at any point is equal to $2px$, where p is a constant and x is the abscissa of the point. That is, the ordinate y of the curve is a function of x whose derivative with respect to x is $\frac{dy}{dx} = 2px$. Therefore, following the method of Arts. 20 and 21 and remembering that $y = 0$ when $x = 0$, we find $y = px^2$. That is the total rise of the curve from $x = 0$ to $x = x$ is px^2. But Fig. 16*b* shows a curve which has the same slope for each value of x as the curve in Fig. 16*a*, except at the place where the vertical jump takes place as indicated by the dotted line j. The rise of the curve in Fig. 16*b* from $x = 0$ to $x = x$ (so as to include the jump at j) cannot be calculated by integration.

27. A definite integral interpreted as the limit of a sum of terms.—Consider a stretched spring of which the elongation is e, the stretching force being equal to ke. If the stretching force is increased very slightly in value the elongation will be increased slightly. Let Δe be the increment of elongation; then the work done by the stretching force in producing the added elongation Δe will be approximately equal to $ke \cdot \Delta e$, the product of the approximate value of the stretching force times the elongation. Let us imagine the work required to produce each separate increment of elongation from $e = 0$ up to any given value of e to be calculated in this way, and let us imagine all these quantities of work to be added together; then we will have an approximate expression for the total work required to produce the given elongation. Using, therefore, $ke \cdot \Delta e$ as a sample of one of these terms we may write:

$$\text{approximate value of } W = \sum_{e=o}^{e=e} ke \cdot \Delta e \qquad (1)$$

where the letter Σ indicates that a large number of terms like the sample term $(ke \cdot \Delta e)$ are to be added together. Now if the successive increments of elongation Δe are chosen smaller and smaller (the number of terms to be added greater and greater) then the sum indicated in equation (1) will approach more and

more to an exact expression for the work W required to produce the given elongation e, and this limiting form of equation (1) is indicated thus:

$$\text{Correct value of } W = \int_{e=o}^{e=e} ke \cdot de \qquad (2)$$

namely the definite integral of $ke \cdot de$ between the limits or boundaries $e = 0$ and $e = e$.

Another example.—Imagine the area A of Fig. 14a to be divided into narrow strips as shown in Fig. 17, the width of each strip being Δx. Consider the shaded strip as shown in the figure, the edge of this strip being at a distance x from the y-axis. Multiplying the width of this strip Δx, by its approximate height y ($= px^2$) we get its approximate area. The approximate area of each of the strips may be calculated in this way, and the sum of such products gives an approximate expression for the total area A of Fig. 14a; and this expression for A becomes more and more nearly exact the greater the number of strips. That is we may write, as above:

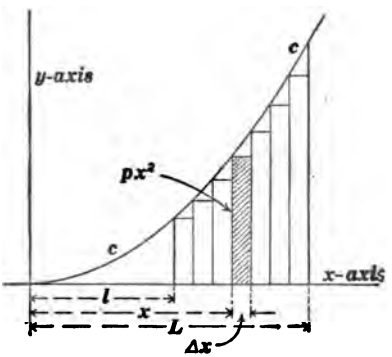

Fig. 17.

$$\text{Approximate value of } A = \sum_{x=l}^{x=L} px^2 \cdot \Delta x$$

and the limiting value of this sum $\Sigma px^2 \cdot \Delta x$ when the number of strips is increased indefinitely is the correct value of A and it is indicated thus:

$$\text{Correct value of } A = \int_{x=l}^{x=L} px^2 \cdot dx$$

28. Plan of this treatise.—Calculus is usually treated in two parts, namely, *differential calculus* and *integral calculus*. Differential calculus is concerned primarily with the finding of derivatives of functions as exemplified in Arts. 6, 8, 10 and 11; and the essence of integral calculus is the finding of the algebraic expression of a function when the derivative of the function is given as exemplified in Arts. 19, 20 and 21.

The application of calculus to any physical problem usually involves three distinct steps. Thus in Art. 20 the following three steps are involved: (*a*) The finding of the derivative $\frac{dW}{de}$ by making use of fundamental principles in physics, (*b*) the becoming familiar with the derivative of ax^2 in Art. 10, and (*c*) the substitution of $\frac{1}{2}k$ for a and e for x so as to get a known function whose derivative is ke

The application of calculus in engineering involves these three steps in nearly every case, and the following development of calculus consists of three parts (more or less intermingled) as follows:

(*a*) The setting up of derivative expressions (differential equations) with the help of fundamental principles of physics.

(*b*) The study of derivatives of algebraic functions; and

(*c*) Practice in transforming derivative expressions to standard forms for purposes of recognition.

The most important branches of calculus which do not fall into this outline are the discussion of maximum and minimum values of functions and the expansion of functions in series.

A table of functions and their derivatives is given in Appendix B. The student should get into the habit of referring to this table.*

PROBLEMS.

Using the rule which is given in Art. 24, verify the following differential equations:

* Or of differential expressions and their integrals.

1. $dy = \frac{1}{2}\sqrt{\frac{3}{x}} \cdot dx$, when $y^2 = 3x$.

2. $dy = \frac{3(ax^2 - 1)}{2\sqrt{ax^3 - 3x}} \cdot dx$, when $y^2 = ax^3 - 3x$.

3. $dy = \frac{4ax + 3}{3(2ax^2 + 3x)^{2/3}} \cdot dx$, when $y^3 = 2ax^2 + 3x$.

4. $dy = -\frac{x}{\sqrt{a^2 - x^2}} \cdot dx$, when $y^2 = a^2 - x^2$.

5. $dy = -\frac{1}{2x^{3/2}} \cdot dx$, when $y^2 = \frac{1}{x}$.

6. $dy = -\frac{1}{3x^{4/3}} \cdot dx$, when $y^3 = \frac{1}{x}$.

The following problems 7 to 10 involve the evaluation of definite integrals as explained in Art. 26 and as exemplified in Arts. 20 and 21.

7. A man earns money steadily at a rate which increases uniformly from zero at the beginning to five dollars per day after 1000 days. Find how much the man earns during the period of 1000 days. Ans. 2500 dollars.

8. A man earns money steadily at a rate which is equal to bt^2 dollars per day, where t is elapsed time in days. When $t = 0$ the wage rate is zero, and when $t = 1000$ days the wage rate is five dollars per day. Find how much the man earns during the period of 1000 days. Ans. 1667 dollars.

9. Consider the parabola $y = 2x^2$. Find the area under this curve between ordinates at $x = 2$ and $x = 10$. Ans. 661.3.

10. Find an expression for the area under the curve $y = qx$ between an ordinate at $x = a$ and the ordinate corresponding to any abscissa x. Ans. $A = \frac{1}{2}qx^2 - \frac{1}{2}qa^2$.

11. The length of the arc s in Fig. $p11$ is a function of the angle θ. Find an expression for $\frac{ds}{d\theta}$, θ being expressed in radians and the equation to the curve being $r = k\theta + b$, where k and b are constants. Ans. $\frac{ds}{d\theta} = \sqrt{k^2 + r^2}$.

Note. When $y = ax^2$ it is easy to find an exact expression for the value of Δy in terms of Δx; but in many cases it is difficult to set up an exact expression for the increment of a function in terms of the increment of the independent variable. In the present case, however, the limiting relation between $\Delta\theta$ and Δs can be seen intuitively. *As the angle $\Delta\theta$ approaches zero the three lines $r\cdot\Delta\theta$, Δr and Δs in Fig. p11 approximate more and more nearly to three straight sides of a right angled triangle.* Therefore the limiting relation between $r\cdot\Delta\theta$, Δr and Δs is $\Delta s = \sqrt{(r\cdot\Delta\theta)^2 + (\Delta r)^2}$.

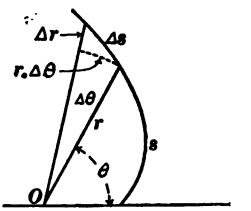

Fig. p11

12. A rotating vessel is filled with liquid of which the density is δ pounds per cubic foot. The speed of the vessel is n revolutions per second or $2\pi n$ radians per second $(= \omega)$. In such a vessel the pressure is a function of r. Find an expression for the derivative $\frac{dp}{dr}$. Ans. $\frac{dp}{dr} = \omega^2 r$.

Note. The argument of this problem furnishes a good example of the method which is referred to in Art. 24. See note to problem 15 on page 233. In this argument distances are expressed in feet and pressures are expressed in poundals per square foot.

CHAPTER II.

FORMULAS FOR DIFFERENTIATION AND INTEGRATION.

29. Differentiation by development and differentiation by rule.—Throughout the previous chapter the derivative of a function (as in Arts. 10 and 11) is determined by developing an expression for $\frac{\Delta y}{\Delta x}$, and considering the limiting form of this expression as Δy and Δx approach zero. This is the fundamental method of differentiation. *It is possible, however, by means of this fundamental method to establish a few rules which enable one to write down the derivative of almost any algebraic function.* It is the object of this chapter to establish these rules and give examples of their use.

Graphical differentiation.—The ordinates of the curve cc in Fig. 18 represent the decreasing temperatures of a vessel of water

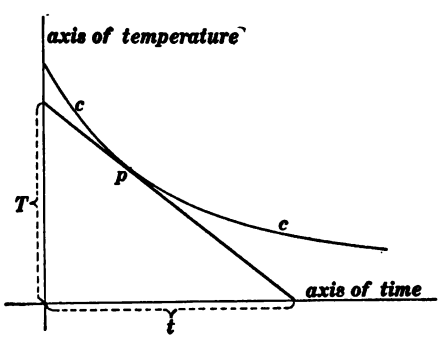

Fig. 18.

which has been heated and set aside to cool, and the abscissas represent elapsed times. To determine the rate of cooling of the vessel at any instant draw a tangent to the curve at the point p which corresponds to the given instant and measure the inter-

cepts T (in degrees) and t (in seconds). Then the quotient T/t gives the desired rate of cooling at the given instant in degrees per second. This operation which involves the plotting of the curve cc from observed data, the laying off of the tangent line, and the measurement of the intercepts T and t, may be called *graphical differentiation.*

Physical differentiation.—The speed of a runner is determined by observing the time for the runner to cover a measured distance. The speed of a rotating dynamo armature is determined by counting the number of revolutions (by means of a revolution counter) which take place in a known interval of time. These determinations may, very properly, be called *physical differentiation.* Thus to determine the rate of supply of water by a spring is to "differentiate" the spring, and the simplest method is to make the differentiation by means of a dipper and a dollar watch.

30. Differentiation of ax^n.—Let

$$y = ax^n \tag{1}$$

Then:

$$\frac{dy}{dx} = nax^{n-1} \tag{2}$$

or

$$dy = nax^{n-1} \cdot dx \tag{3}$$

Equations (2) and (3) are true for any value of n, but n is understood to be a positive integer in the following proof.*

Proof.—Writing $y + \Delta y$ for y and writing $x + \Delta x$ for x in equation (1) we have:

$$y + \Delta y = a(x + \Delta x)^n \tag{4}$$

When n is a positive integer $(x + \Delta x)^n$ can be expanded by the binomial theorem,† and we have

* A general proof is given in Art. 39.

† The binomial theorem is:

$$(a + b)^n = a^n + n\frac{a^{n-1}b}{1} + n(n-1)\frac{a^{n-2}b^2}{1\cdot 2} + n(n-1)(n-2)\frac{a^{n-3}b^3}{1\cdot 2\cdot 3} + \text{etc.}$$

$$y + \Delta y = ax^n + nax^{n-1} \cdot \Delta x + \frac{n(n-1)ax^{n-2}(\Delta x)^2}{1 \cdot 2} + \text{etc.} \quad (5)$$

Subtracting equation (1) from equation (5), member by member, we have:

$$\Delta y = nax^{n-1} \cdot \Delta x + \frac{n(n-1)ax^{n-2}(\Delta x)^2}{1 \cdot 2} + \text{etc.} \quad (6)$$

Whence, proceeding exactly as in Arts. 10 and 11, we get equation (2).

31. Differentiation of the sum of two or more functions.—Let u and v be any two functions of x, and let

$$\mathbf{y = u + v} \quad (1)$$

Then

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\mathbf{du}}{\mathbf{dx}} + \frac{\mathbf{dv}}{\mathbf{dx}} \quad (2)$$

or

$$\mathbf{dy = du + dv} \quad (3)$$

For example, if $u = ax^2$ and $v = bx^3$, then $\frac{du}{dx} = 2ax$ and $\frac{dv}{dy} = 3bx^2$; and the derivative of $u + v$ is the sum of the derivatives $\frac{du}{dx}$ and $\frac{dv}{dx}$; or the differential of $(u + v)$ is the sum of the two differentials $2ax \cdot dx$ $(= du)$ and $3bx^2 \cdot dx$ $(= dv)$.

The fact which is expressed by equation (2) is self-evident when it is expressed in familiar terms. Two men A and B contribute continuously to a fund y, and the sums contributed by A and B up to a given date are u dollars and v dollars, respectively; and of course $y = u + v$. Now $\frac{du}{dt}$ is the rate at which A contributes, and $\frac{dv}{dt}$ is the rate at which B contributes; and the rate of growth of the fund, $\frac{dy}{dt}$, is equal to $\frac{du}{dt} + \frac{dv}{dt}$! The essential part of this proposition is equivalent to the following statement: if one has an investment income of 10,000 dollars per year and

a salary income of 250 dollars per year, his total income is 10,250 dollars per year!*

32. Differentiation of the product of two functions.—Let u and v be any two functions of x, and let

$$\mathbf{y = uv} \tag{1}$$

Then

$$\mathbf{\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}} \tag{2}$$

or

$$\mathbf{dy = u \cdot dv + v \cdot du} \tag{3}$$

Proof.—Writing $y + \Delta y$ for y, writing $u + \Delta u$ for u, and writing $v + \Delta v$ for v in equation (1) we have:

$$y + \Delta y = uv + u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v \tag{4}$$

whence, subtracting equation (1) from equation (4) member by member, we have:

$$\Delta y = u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v \tag{5}$$

so that

$$\frac{\Delta y}{\Delta x} = u \cdot \frac{\Delta v}{\Delta x} + v \cdot \frac{\Delta u}{\Delta x} + \frac{\Delta u \cdot \Delta v}{\Delta x} \tag{6}$$

Now u and v are supposed to be continuous functions of x so that $\frac{\Delta u}{\Delta x}$ and $\frac{\Delta v}{\Delta x}$ both approach definite finite values as Δx approaches zero,† and consequently $\frac{\Delta u \cdot \Delta v}{\Delta x}$ approaches zero.

* The *necessary* use of unfamiliar symbols in a treatise on calculus tends to obscure the important fact that calculus is largely a phase of an old fashioned branch of mathematics which was called *mental arithmetic*, and which is unfortunately now quite obsolete. Let the student read again Arts. 19, 20 and 21, and he will understand that *integral calculus* is mental arithmetic pure and undefiled. Indeed, the fundamental argument of integral calculus, as developed in Arts. 19, 20 and 21, cannot be expressed algebraically; and to one who understands this argument the algebraic formulation of it in Arts. 26 and 27 is merely an annoyance.

† No function y can be differentiated when $\frac{\Delta y}{\Delta x}$ does not approach a definite finite limiting value as Δx approaches zero, and in the study of differ-

Therefore equation (2) is obtained as the limiting form of equation (6).

Example.—Let $u = ax^2 + bx$, let $v = cx^3 + 1$ and let $y = (ax^2 + bx)(cx^3 + 1)$. To differentiate this expression substitute the values of u and v, and the values of

$$du\ (= 2ax \cdot dx + b \cdot dx) \quad \text{and} \quad dv\ (= 3cx^2 \cdot dx)$$

in equation (3), and we have

$$dy = (ax^2 + bx) \times 3cx^2 \cdot dx + (cx^3 + 1)(2ax + b) \cdot dx$$

33. Differentiation of the quotient of two functions.—Let u and v be any two functions of x, and let

$$y = \frac{u}{v} \tag{1}$$

Then

$$\frac{dy}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2} \tag{2}$$

or

$$dy = \frac{v \cdot du - u \cdot dv}{v^2} \tag{3}$$

Proof.—From equation (1) we have

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v} \tag{4}$$

ential calculus we are concerned only with functions which can be differentiated. Indeed, in all strictness, algebraic functions are the only functions which can be differentiated. Let Δm be the amount of material (in grams) in volume Δv. The ratio $\frac{\Delta m}{\Delta v}$ does not approach a definite limiting value as Δv approaches zero, because material is made up of atoms. Indeed most natural functions are essentially discontinuous, and therefore they cannot in strictness be differentiated; their discontinuous character, however, is nearly always ignored in ordinary applications of mathematical theory. There are certain algebraic functions also which cannot be differentiated. For example, the expressions for y in the answers to problems 1, 2, 3, 4, 5 and 6 at the end of Art. 112 cannot be differentiated.

whence, subtracting equation (1) from equation (4) member by member, we have

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} \tag{5}$$

Reducing $\frac{u + \Delta u}{v + \Delta v}$ and $\frac{u}{v}$ to a common denominator, equation (5) becomes:

$$\Delta y = \frac{v \cdot \Delta u - u \cdot \Delta v}{v^2 + v \cdot \Delta v} \tag{6}$$

Therefore we have:

$$\frac{\Delta y}{\Delta x} = \frac{v \cdot \frac{\Delta u}{\Delta x} - u \cdot \frac{\Delta v}{\Delta x}}{v^2 + v \cdot \Delta v} \tag{7}$$

and this equation reduces to equation (2) when Δx, Δu and Δv approach zero.

Note. Go to top of page 42 of the 1913 edition of Franklin, MacNutt and Charles' *Calculus*.

LIST OF CORRECTIONS TO OLD EDITION OF FRANKLIN, MacNUTT & CHARLES' CALCULUS.

(Kindness of Professor H. S. Uhler.)

Page 21. Problem 7. The answer should be 10 miles per hour.
Problem 9, third line. For *growing* read *decreasing*.

Page 30. Problem 2. Answer incorrect. Insert coefficient 2 in the denominator.
Problem 3. Answer incorrect. Insert coefficient 3 in the denominator.
Problem 10. Answer should be 312 $\pi^3 n^2 r^3$.

Page 35. Line 6. The second member of equation (6) should be the integral of $ke \cdot de$ between *zero* and *e*.

Page 48. Denominator in second member of equation 13 should be log *a*.

Page 60. The discussion on page 60 is mixed up because of the use of sine instead of cosine.

Page 66. Line 1 should read $R = 1/2$ instead of $R = 2$.

Page 69. Problem 2. The second answer should be 9 inches.
Problem 3. Insufficient data are given.

Page 104. Problem 7. Answer should be 1017.9 cubic inches.
Problem 8. Answer should be 6425.3 cubic inches.

Page 108. Problem 3. Answer should be 131.7 square inches.

Page 111. Problem 6. Data incorrect.

Page 113. Line 20. Read *inversely* for *universally*.

Page 114. Problem 12. Answer should be 8378 cubic inches.

Page 126. In the last sentence read *Fig. 66* for *Fig. 67*.

Page 131. Problem 9. Answer should be 14.54 inches.
Problem 12. Answer should be 9.72 inches.

Page 132. Problem 13. Answer should be 6.93 inches.

Page 140. Problem 3. Line 3. For $r = 6$ *feet* read $r = 3$ *feet*.

Page 149. Problem 4. Answer should be 1855 pound-feet-squared.

Page 151. Problem 11. Answer should be 10.2 pound-feet-squared.

Page 157. Line 7 of article headed Taylor's series should read $x + h$ for $y + h$. .

Page 165. Second line from bottom, read *25* for *24*.

Page 166. Problem 2. Answer is wrong.

Page 182. Equation (4) read CL for L/C, and change the following work accordingly.

Page 207. Problem 2. The coefficient before the parentheses should be $4a/\pi$ instead of $8a/\pi^2$.

Problem 3. Ditto.

APPENDIX A.

Page 3. The answer to Problem 26 is wrong. The last parenthesis in the answer should have an exponent 4.

Problem 27. Answer (*c*) should be plus 20.

Page 4. The last term in the answer of problem 1 should be *plus* instead of *minus*.

Page 5. Read $2/3b$ for $2b/3$ in Problem 8.

Page 10. Omit dimensions of figure in answer to problem 23.

Page 11. Read cot. for cos. in answer to Problem 13.

Page 16. Read 1/10 for 1/2 in answer to Problem 31.

Read 1/20 for 1/4 in answer to Problem 32.

Answer to Problem 36 is inches not square inches.

Page 19. Denominator should be a^2x instead of a^2x^2 in answer to Problem 9.

Page 20. Coefficient of the last parenthesis should be $-\frac{3a}{2b^2}$ instead of $-\frac{3}{2b^2}$ in answer to Problem 2.

The answer to Problem 4 should be multiplied by $3a^2$.

110*39.

110*39.